Review for Final – Solutions[1]

Assigned: May 11, 2021

Multivariable Calculus MATH 53
with Professor Stankova

Contents

WARNING: The solutions written here are often quite abbreviated. For instance, actually evaluating many of the integrals is done without showing any work. On the final exam, it is extremely important that you include all of the steps that you took when solving the problem. Maybe there was some clever shortcut you saw, but we cannot read your mind, so tell us what you are doing. In general, try to include as many complete English sentences as possible to explain what you are thinking/doing. The solutions given here are much shorter than we expect you to provide.

0.1 Multiple Integrals

1. True/False practice:
   a) For a continuous function f, the value $\iint_R f(x,y)\,dA$ equals the area of the region R.

      Solution: False. If we just had the constant function $f(x,y) = 1$, this would be true, but in general this calculates something like a weighted' area.
   b) For a continuous function f, the value $\iint_R f(x,y)\,dA$ equals the average value of f on the region R.

      Solution: False. We can think of this double integral as calculating the "total" amount of f over the region. To get the average, we'd have to divide off by the size of R.
   c) If $f(x,y)$ is an odd function, then $\iint_R f(x,y)\,dA = 0$.

      Solution: False. For one, it maybe doesn't make total sense to talk about an odd function in more than one variable; we also need to region to be symmetric. What is true is that: if $f(x,y)$ is odd *with respect to one of the variables* (meaning that fixing one variable as constant leaves us with an odd, single-variable function) and R is symmetric with respect to that variable (so for instance if f is odd when we fix y to be a constant, R being symmetric would mean it is the same when reflected across the line $x = 0$), then the integral is zero.

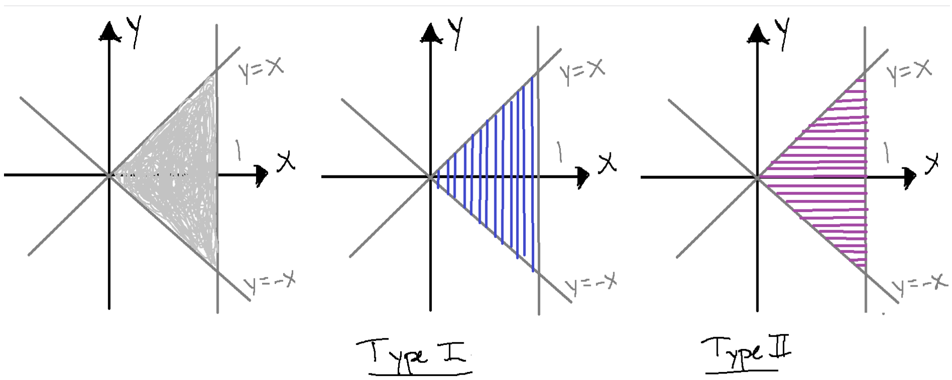
d) Every region in the plane is either Type I or Type II.

Solution: False. There are many regions (like the four-leaf clover) that are neither Type I nor Type II. However, in most cases, we can break the region into smaller pieces that are either Type I or Type II and then evaluate the integral piece by piece.

2. For the given region R, set up the integral $\iint_R f(x,y)\,dA$ for both orders of integration.

a) $R = \{(x,y) \mid 0 \le x \le 1, -x \le y \le x\}$.

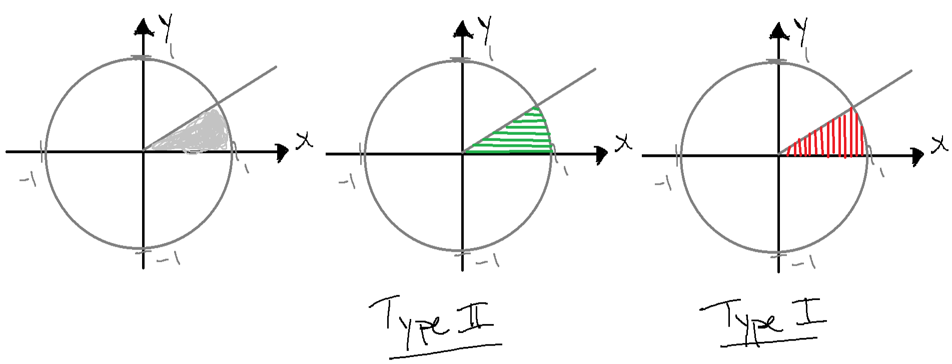
Solution: The easiest way to do this sort of question is to start by drawing the region.



$\int_{x=0}^{x=1} \int_{y=-x}^{y=x} f(x,y)\,dy\,dx$ or $\int_{y=-1}^{y=0} \int_{x=-y}^{x=1} f(x,y)\,dx\,dy + \int_{y=0}^{y=1} \int_{x=y}^{x=1} f(x,y)\,dx\,dy$.

b) R is the sector of the unit circle between the x-axis and $30°$ counterclockwise from the x-axis.

Solution: The easiest way to do this sort of question is to start by drawing the region.



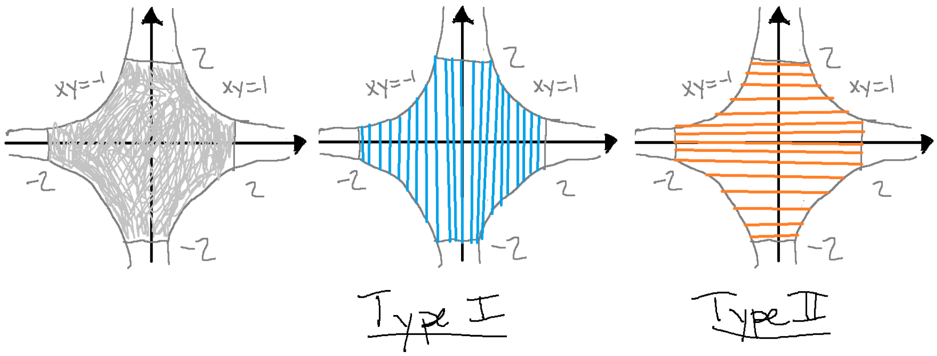
Note the line that cuts out this sector is given by $y = \frac{x}{\sqrt{3}}$. Then we get:

$$\int_{y=0}^{y=\frac{1}{2}} \int_{x=\sqrt{3}y}^{x=\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

$$\int_{x=0}^{x=\frac{\sqrt{3}}{2}} \int_{y=0}^{y=\frac{x}{\sqrt{3}}} f(x,y)\,dy\,dx + \int_{x=\frac{\sqrt{3}}{2}}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

c) R is bounded by the lines $x = 2$, $x = -2$, $y = 2$, $y = -2$, $xy = 1$ and $xy = -1$.

Solution: The easiest way to do this sort of question is to start by drawing the region.



This region is going to have three pieces no matter which way we write it.

$$\int_{-2}^{-\frac{1}{2}} \int_{\frac{1}{x}}^{-\frac{1}{x}} f(x,y)\,dy\,dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-2}^{2} f(x,y)\,dy\,dx + \int_{\frac{1}{2}}^{2} \int_{-\frac{1}{x}}^{\frac{1}{x}} f(x,y)\,dy\,dx$$

$$\int_{-2}^{-\frac{1}{2}} \int_{\frac{1}{y}}^{-\frac{1}{y}} f(x,y)\,dx\,dy + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-2}^{2} f(x,y)\,dx\,dy + \int_{\frac{1}{2}}^{2} \int_{-\frac{1}{x}}^{\frac{1}{x}} f(x,y)\,dx\,dy$$

3. Find the average value of $f(x,y) = x^2y$ over the triangle whose end points are $(0,0)$, $(1,2)$ and $(2,1)$.

   Solution: The average value is given by $\frac{1}{|R|}\iint_R f\,f(x,y)\,dA$. So we need to calculate the integral of this function over the region and then calculate the area of the region. To integrate over this region, we will need to break it up into two pieces no matter how we slice it.

$$\begin{aligned}
\iint_R x^2y\,dA &= \int_0^1 \int_{\frac{y}{2}}^{2y} x^2y\,dx\,dy + \int_1^2 \int_{\frac{y}{2}}^{3-y} x^2y\,dx\,dy \\
&= \int_0^1 \frac{1}{3}x^3y\Big|_{\frac{y}{2}}^{2y}\,dy + \int_1^2 \frac{1}{3}x^3y\Big|_{\frac{y}{2}}^{3-y} \\
&= \int_0^1 \left(\frac{1}{3}8y^4 - \frac{1}{24}y^4\right)\,dy + \int_1^2 \left(\frac{1}{3}(3-y)^3y - \frac{1}{24}y^4\right)\,dy \\
&= \left(\frac{8}{15}y^5 - \frac{1}{120}y^5\right)\Big|_0^1 + \left(\frac{101}{20} - \frac{31}{120}\right) = \frac{638}{120}
\end{aligned}$$

   But then we need to divide this by the area of the triangle. We could calculate $\iint_R a\,dA$ to find this area. Or we could review a long ago fact about vectors. The area of the parallelogram spanned by $\langle 1,2,0\rangle$ and $\langle 2,1,0\rangle$ is equal to the length of cross product of these two vectors (this is important for surface integrals). So the area of the triangle is half this length.]

$$\langle 1,2,0\rangle \times \langle 2,1,0\rangle = \langle 0,0,-3\rangle$$

   So the area of the region is $\frac{3}{2}$, and our final answer is $\frac{638}{120}\cdot\frac{2}{3}$.

4. Suppose D is the region in $\mathbb{R}^3$ given by: $x \geq 0$, $y \geq 0$, $z \geq 0$, $x \leq y$ and $x+y+z \leq 1$. Find the volume of this region using a triple integral.

Solution: To find volume, we want to calculate $\iiint_D 1\,dV$ where D is the given region. Note that this region is a tetrahedron with corners at $(0,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(0.5,0.5,0)$. There are many different ways to break this region up as an iterated integral, to get the bounds. But one way keeps us from having to split the integral into multiple pieces.

Working from the inside out. For any possible value of x,y the z-coordinate has the same bounds: $0 \leq z \leq 1-x-y$. Then, no matter what value of x or z we are at, we have $x \leq y$ and we have that $y \leq 1-x$. Finally, x can be any value from 0 to $\frac{1}{2}$. So putting it together, we get:

$$\begin{aligned}
\iiint_D 1\,dV &= \int_0^{\frac{1}{2}} \int_x^{1-x} \int_0^{1-x-y} 1\,dz\,dy\,dx \\
&= \int_0^{\frac{1}{2}} \int_x^{1-x} 1-x-y\,dy\,dx \\
&= \int_0^{\frac{1}{2}} \left(y - xy - \frac{1}{2}y^2 \right)\Bigg|_x^{1-x} dx \\
&= \int_0^{\frac{1}{2}} \left(1-x-x(1-x)-\frac{1}{2}(1-x)^2\right) - \left(x-x^2-\frac{1}{2}x^2\right) dx \\
&= \int_0^{\frac{1}{2}} 2x^2-2x+\frac{1}{2}\,dx \\
&= \frac{1}{12} - \frac{1}{4} + \frac{1}{4} = \frac{1}{12}
\end{aligned}$$

0.2 Change of Variables

1. True/False practice:
   a) When evaluating $\int_{-1}^1 f(x)\,dx$, if we use the u-substitution $u = x^2$, we can see that the integral $\int_{-1}^1 f(x)\,dx = \int_1^1 f(\sqrt{u})\frac{1}{2\sqrt{u}}\,du = 0$, since the upper and lower bound are the same.

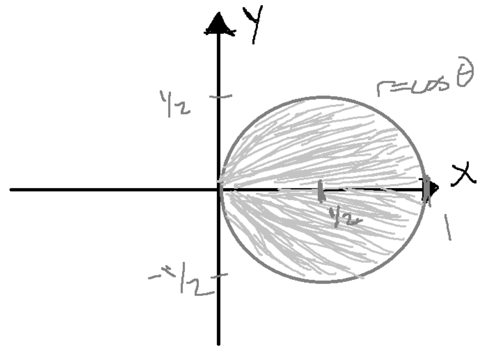
      Solution: False. This looks almost valid, and is an easy trap to fall into in single-variable calculus. The problem here is that our transformation $u = x^2$ is *not one-to-one* on the original interval $[-1,1]$. This interval gets sent to $0 \leq u \leq 1$, However, each point inside this interval is hit twice (both -0.5 and 0.5 are sent to 0.25). Really, we would have to separate the original interval into two pieces $[-1,0]$ and $[0,1]$. Then the transformation looks like $x = -\sqrt{u}$ on the first interval and $x = \sqrt{u}$ on the second interval.

   b) By changing to spherical coordinates, we can calculate the volume of the unit sphere as $\int_0^1 \int_0^{2\pi} \int_0^{2\pi} \rho^2 \sin\phi\,d\theta\,d\phi\,d\rho$.

      Solution: False. This is almost correct, just one of the bounds in the integral is incorrect. When we change to spherical coordinates, φ measures the angle from the positive z-axis (from the north pole). Since θ is already allowed to range form 0 to 2π, we only want φ to range from 0 to π, as otherwise we will start to hit the same point multiple times (we'd start circling back up from the south pole).

2. Sketch the region bounded by $r = \cos\theta$ where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Find the integral of $f(x,y) = \frac{1}{x}$ on this region.

Solution: If we change this curve from polar to Cartesian coordinates, it will be easier to draw. If we multiply both sides of the equation by r, we get $r^2 = r\cos\theta$. Then in Cartesian coordinates, this becomes $x^2 + y^2 = x$. So finally, we can complete the square to get the familiar equation $\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$. As θ ranges from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, r will range from 0 to 1 and then back to 0.



Then we want to find $\iint_R \frac{1}{x}\,dA$. The easiest way to do this is by changing everything to polar coordinates. Then the function becomes $\frac{1}{r\cos\theta}$ and $dA = r\,dr\,d\theta$. The region is already described above as $0 \le r \le \cos\theta$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. So we get the integral:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} \frac{1}{\cos\theta}\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,d\theta = \pi$$

3. Suppose D is the disc centered at the origin with radius a. What is the average distance from the origin of a point on this disc?

   Solution: We are trying to calculate the average of some function over a region; in particular $f(x, y) = \sqrt{x^2 + y^2}$ is the distance from the origin. This can be done by finding $\frac{1}{|D|}\iint_D f(x, y)\,dA$. The size of our region is simply the area of the disc with radius a, which is $|D| = \pi a^2$. So then we just need to calculate:

$$\frac{1}{\pi a^2}\iint_D \sqrt{x^2 + y^2}\,dA$$

   This can most easily be done by converting to polar coordinates. Then our region D becomes $0 \le \theta 2\pi$ and $0 \le r \le a$, and dA becomes $r\,dr\,d\theta$. Then plugging this in gives us:

$$\frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a r \cdot r\,dr\,d\theta = \frac{1}{\pi a^2}\cdot 2\pi \frac{r^3}{3}\Bigg|_0^a = \frac{2}{3}a$$

4. Suppose R is the parallelogram whose corners are $(-1, -1)$, $(1, 0)$, $(0, 1)$, and $(2, 2)$. Find a change of variables that transforms the region to the unit square $[0, 1] \times [0, 1]$. Use this change of variables to evaluate $\iint_R x^2\,dA$.

   Solution: The bottom line for our original parallelogram is given by $2y = x - 1$ and the top line is given by $2y = x + 2$. We want these lines to get sent to $v = 0$ and $v = 1$ respectively. We can rearrange these two equations as $x - 2y - 1 = 0$ and $x - 2y - 1 = -3$. So to get $v = 0$

and $v = 1$, we can let $-3v = x - 2y - 1$.

We can do the same thing to find $u = 0$ and $u = 1$. The left line is given by $2x - y + 1 = 0$ and the right line is given by $2x - y + 1 = 3$. So we can let $3u = 2x - y + 1$. We can then solve for x and y to get our change of coordinates:

$$x = 2u + v - 1 \qquad y = u + 2v - 1$$

Then we want to find the Jacobian in order to turn $dx\,dy$ into $du\,dv$.

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

So we can set up our final integral:

$$\iint_R x^2\,dA = \int_0^1 \int_0^1 (2u + v - 1)^2 \cdot 3\,du\,dv = 3 \cdot \frac{2}{3}$$

5. Find $\int_0^1 \int_0^1 \frac{x-y}{x+y}\,dA$ using the change of variables $u = x - y$ and $v = x + y$.

   Solution: First we can solve for x and y in terms of u and v. This gives us $x = \frac{1}{2}u + \frac{1}{2}v$ and $y = -\frac{1}{2}u + \frac{1}{2}v$. Then we can determine what our new region is after the transformation. We started with $0 \leq x \leq 1$, so this becomes $0 \leq \frac{1}{2}u + \frac{1}{2}v \leq 1$. Then solving for v gives us $-u \leq v \leq 2 - u$. Similarly, we had $0 \leq y \leq 1$, which becomes $0 \leq -\frac{1}{2}u + \frac{1}{2}v \leq 1$. Again, solving for $u \leq v \leq 2 + u$. Altogether, this gives us the square in the uv-plane with corners $(0, 0), (1, 1), (0, 2), (-1, 1)$, call this new region S.

   Then to start evaluating this integral, we also need the *Jacobian*. This is calculated by:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} - \frac{-1}{4} = \frac{1}{2}$$

   Then we get the integral $\iint_S \frac{u}{v}\frac{1}{2}\,dA$. We can then use our basic knowledge of double integrals to evaluate this (although we'd need to break the region into two pieces). However, we can also use a bit of symmetry to simplify everything. In particular, for any fixed value of v, the function we are integrating $\frac{u}{2v}$ is *odd*. Additionally, the region is symmetric over the v-axis. So the final integral will just be 0.

6. The density of a block of metal is given by $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2}}$. If you cut out a solid by taking the top half of the unit sphere centered at the origin, what is the total mass of the piece you end up with?

   Solution: To get the total mass, we will want to integrate over the region. In particular, we want to find $\iiint_R \frac{1}{\sqrt{x^2+y^2}}\,dV$. This is most easily done in spherical coordinates, especially since our region is just the top hemisphere. So we have $x = \rho\cos\theta\sin\varphi$, $y = \rho\sin\theta\sin\varphi$, and $z = \rho\cos\varphi$. The top half of the unit sphere is given by $0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \frac{\pi}{2}$.

Finally, dV becomes $\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$. So our integral is:

$$\begin{aligned}
\iiint_R \frac{1}{\sqrt{x^2+y^2}} \, dV &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{\sqrt{\rho^2 \sin^2\varphi}} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 \rho \, d\rho \, d\varphi \, d\theta \\
&= 2\pi \cdot \frac{\pi}{2} \frac{\rho^2}{2} \bigg|_0^1 = \frac{\pi^2}{2}
\end{aligned}$$

0.3 Vector Fields

1. True/False practice:
   a) The gradient of the gradient of $f(x,y)$ is always equal to 0.

      Solution: False. This is a meaningless sentence! The gradient of $f(x,y)$ will be a vector function; $\nabla f : \mathbb{R}^2 \to \mathbb{R}^2$. But then we can't take the gradient of a vector function. There are other operations that serve a similar purpose, but gradient itself only makes sense for scalar functions.

   b) If $\vec{F}(x,y)$ is a vector field, there is *at most* one function f so that $\nabla f = \vec{F}$.

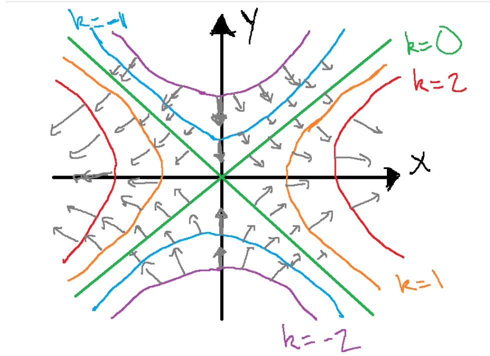
      Solution: False. There are always lots of different potential functions. Suppose $f(x,y)$ is a potential function for $\vec{F}(x,y)$. Then it will also be true that $f(x,y) + 17.549832$ is a potential function (or any other constant you could add).

   c) The gradient of $f(x,y)$ always points parallel to the level curves of $f(x,y)$.

      Solution: False. The gradient will always point *perpendicular* to the level curve. This is an old fact from the previous midterm (but it's still important!). One way to think about it is in terms of directional derivatives. If $\vec{r}(t) = k$ is a level curve for $f(x,y)$, then $\vec{r'}(t)$ will give us tangent vectors to this curve. The directional derivative along the level curve will be given by $\nabla f \circ \vec{r'}$. But along the level curve, the function is constant, so this directional derivative must be zero. Since the dot product is then zero, these vectors are perpendicular.

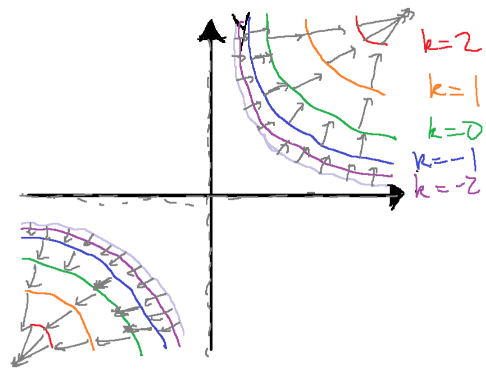
2. For the given function f, find the domain, sketch 5 level curves and sketch ∇f.
   a) $f(x,y) = x^2 - y^2$.

      Solution: The domain is all of $\mathbb{R}^2$. We have $\nabla f = \langle 2x, -2y \rangle$. The level curves are given by the hyperbolas $x^2 - y^2 = k$ for any value of k. Note that when $k = 0$, this simplifies to $(x+y)(x-y) = 0$, which is just two lines. The gradient is perpendicular to the level curve, pointing in the increasing direction.



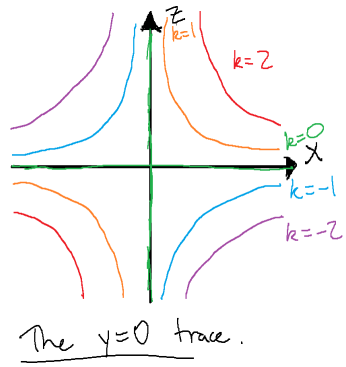
b) $f(x, y) = \ln xy$.

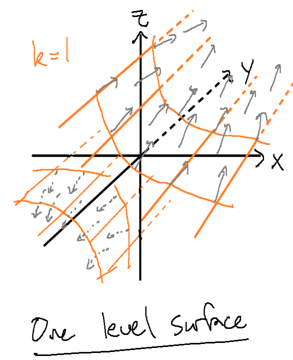
Solution: We can't take the logarithm of something less than or equal to 0, so the domain is $xy > 0$. Said in a better way, the domain is the first quadrant or the third quadrant (not including the axes). The level curves are given by $\ln xy = k$ for any value of k, i.e. $xy = e^k$. These are again hyperbolas, but note that we never get the two straight lines $xy = 0$. The gradient is $\nabla f = \langle y, x\rangle$, these arrows all point towards the line $y = x$ and get longer as we get farther from the origin.



c) $f(x, y, z) = xz$.

Solution: The domain is all of $\mathbb{R}^3$. This is a function from $\mathbb{R}^3 \to \mathbb{R}$, so the "level curves" are actually surfaces in three-dimensional space. Each of these surfaces $xz = k$ (where k can be anything) is actually a special type of surface we have talked about before, a "cylinder". In particular, if we draw the surface on the trace $y = 0$, all of the other traces are parallel to this. The gradient is $\nabla f = \langle z, 0, x\rangle$, which are vectors that are perpendicular (normal) to the level surfaces.





3. If $f(0,0) = 2$ and $\nabla f(x,y) = \langle e^{xy} - y^2 + x + 2, e^{xy} - 2yx + 3\rangle$, estimate $f(0.1, 0.1)$.

Solution: The linear estimation of a function around the point $(0,0)$ is given by:

$$\begin{aligned} f(x,y) &\approx f(0,0) + \nabla f(0,0) \circ \langle x - 0, y - 0\rangle \\ &\approx 2 + \langle 3, 4\rangle \circ \langle x, y, \rangle \end{aligned}$$

So we get that $f(0.1, 0.1) \approx 2 + (0.3 + 0.4) = 2.7$.

4. For the following vector fields, determine whether they are *conservative*. If it is conservative, find a potential function.
   a) $\vec{F}(x,y) = \langle x^2, y^2\rangle$.

      Solution: Since the domain of the function is all of $\mathbb{R}^2$, we can check whether or not it is conservative by looking at whether $Q_x - P_y = 0$ (the mixed partials are equal). In this case, $Q_x = 0$ and $P_y = 0$, so this is conservative. Taking two partial antiderivatives gives us a possible potential function of $f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$.
   b) $\vec{F}(x,y) = \langle y^2, x^2\rangle$.

      Solution: If we check the mixed partial derivatives again, we get $Q_x = 2x$ and $P_y = 2y$. Since these are not equal, this function is not conservative.
   c) $\vec{F}(x,y,z) = \langle 8xyz^3, 4x^2z^3 + 2yz, 12x^2yz^2 + y^2 + 1\rangle$.

      Solution: Again the domain is all of $\mathbb{R}^3$, so to check whether it is conservative, we want to check whether all of the mixed partial derivatives agree. The easiest way is noting that this is just checking whether $\nabla \times \vec{F} = \langle 0,0,0\rangle$.

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} 1^{st} & 2^{nd} & 3^{rd} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 8xyz^3 & 4x^2z^3 + 2yz & 12x^2yz^2 + y^2 + 1 \end{vmatrix} \\ &= \langle 12x^2z^2 + 2y - 12x^2z^2 - 2y, 24xyz^2 - 24xyz^2, 8xz^3 - 8xz^3\rangle = \langle 0,0,0\rangle \end{aligned}$$

      So this vector function is conservative, and we can try to find a potential function by taking antiderivatives. For instance, from the first component, we can take the x-antiderivative, which will give us $f(x,y,z) = 4x^2yz^3 + c(y,z)$, where $c(y,z)$ is some unknown function that is constant with respect to x.

      Then looking at the second component, we can take the y-derivative of our current guess and compare that to what it is supposed to be $f_y = 4x^2z^3 + c_y(y,z) = 4x^2z^3 + 2yz$. So if we want $c_y(y,z) = 2yz$, we can take $c(y,z) = y^2z + d(z)$, where d is some unknown function that is constant with respect to x and y.

Finally, we can look at the last component. Our current guess is $f(x,y,z) = 4x^2yz^3 + y^2z + d(z)$. Comparing the z-derivative of this to what it is supposed to be gives $f_z = 12x^2yz^2 + y^2 + d'(z) = 12x^2yz^2 + y^2 + 1$. So for $d'(z) = 1$, we end up with $d(z) = z + C$, where C is just a number. Putting it all together gives us a potential function: $f(x,y,z) = 4x^2yz^3 + y^2z + z + C$.

d) $\vec{F}(x,y,z) = \langle 6x^2yz, 2x^3z, 3x^3y\rangle$.

Solution: We follow the same procedure as above, however the curl is not equal to zero. In particular, if we look at the mixed partials: $\frac{\partial}{\partial z}2x^3z \neq \frac{\partial}{\partial y}3x^3y$, so this is not conservative.

0.4 Line Integrals

1. True/False practice:

a) If we can find two different paths between the same points that give us different values for $\int_C \vec{F} \circ d\vec{r}$, then we can conclude that $\vec{F}$ is not conservative.

Solution: True. For any conservative vector field, the value of $\int_C \vec{F} \circ d\vec{r}$ is determined by the ending and starting point of C. So if there are two paths that have the same start point and the same end point, a conservative vector field would cause the integrals to be equal.

b) If C is a curve between $(1,1)$ and $(2,3)$ then $\int_C f\,ds$ will change signs depending on which point we choose as the start or end of our parameterization.

Solution: False. There is an important difference between line integrals of scalar functions and vector functions. For a scalar function, we will always get the same answer regardless of parameterization (the arc length isn't going to change if we travel the opposite direction). However, the line integral of a vector function can change signs depending on which is the start/end point (the orientation).

c) If we parameterize the curve C by moving twice as fast as someone else, then our $\int_C \vec{F} \circ d\vec{r}$ will be twice as big as theirs.

Solution: False. The only thing that could change with our choice of parameterization in the case of the line integral of a vector function is the orientation, changing the sign.

2. Find $\int_C f\,ds$.

a) $f(x,y) = x^2y$ and C is the line from $(0,0)$ to $(2,1)$.

Solution: First, we want to parameterize the line connecting our two points. Note the line is $y = \frac{1}{2}x$, so we could take the parameterization $\vec{r}(t) = \langle t, \frac{1}{2}t\rangle$ where t goes from 0 to 2. Then we want to calculate $ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt = \sqrt{1 + \frac{1}{4}}\,dt = \frac{\sqrt{5}}{2}\,dt$. Now, we can start putting everything together in our integral.

$$\int_C x^2 y\, ds = \int_0^2 t^2 \cdot \frac{1}{2} t \frac{\sqrt{5}}{2}\, dt$$
$$= \frac{\sqrt{5}}{4} \int_0^2 t^3\, dt = \sqrt{5}$$

b) $f(x,y) = ye^x$ and C is the bottom half of the unit circle between $(-1,0)$ and $(1,0)$.

Solution: First, we want to parameterize the curve connecting our two points. We could take the parameterization $\vec{r}(t) = \langle \cos t, \sin t \rangle$ where t goes from π to 2π. Then we want to calculate $ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt = \sqrt{\sin^2 t + \cos^2 t}\, dt = dt$. Now, we can start putting everything together in our integral.

$$\int_C ye^x\, ds = \int_\pi^{2\pi} \sin t e^{\cos t}\, dt$$
$$= -e^{\cos t}\Big|_\pi^{2\pi} = -e^1 + e^{-1}$$

c) $f(x,y,z)$ is the distance from the origin and C is the helix given by $\langle \cos \pi t, \sin \pi t, t \rangle$ from $(1,0,0)$ to $(-1,0,1)$.

Solution: We already have a parameterization for our curve, almost. We also need to know what t ranges between. In this case, we can just look at the z-coordinate to see that $0 \le t \le 1$. Then we want to calculate $ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt = \sqrt{\pi^2 \sin^2 \pi t + \pi^2 \cos^2 \pi t + 1}\, dt = \sqrt{2\pi^2 + 1}\, dt$. Now, we can start putting everything together in our integral.

$$\int_C \sqrt{x^2 + y^2 + z^2}\, ds = \int_0^1 \sqrt{\cos^2 \pi t + \sin^2 \pi t + t^2}\sqrt{2\pi^2 + 1}\, dt$$
$$= \sqrt{2\pi^2 + 1} \int_0^1 \sqrt{1 + t^2}\, dt$$

3. Find $\int_C \vec{F} \circ d\vec{r}$.

a) $F(x,y) = \langle x^2, 2xy \rangle$ and C is the line from $(-1,1)$ to $(2,3)$.

Solution: First, we can check whether this is conservative. if it is, we can use the FTL to save a lot of time. Unfortunately, the mixed partial derivatives are not equal, so we have to do it the brute force way. We want to parameterize the line connecting our two points. We can easily take a parameterization that starts at $(-1,1)$ when $t = 0$ and moves to $(2,3)$ when $t = 1$ by $\vec{r}(t) = \langle 2t - (1-t), 3t + (1-t) \rangle = \langle 3t - 1, 2t + 1 \rangle$. Then we want to calculate $d\vec{r} = \vec{r'}(t)\, dt = \langle 3, 2 \rangle\, dt$. Now, we can start putting everything together in our integral.

$$\begin{aligned}\int_C \langle x^2, 2xy\rangle \circ d\vec{r} &= \int_0^1 \langle (3t-1)^2, 2(3t-1)(2t+1)\rangle \circ \langle 3,2\rangle\, dt \\ &= \int_0^1 3(3t-1)^2 + 4(3t-1)(2t+1)\, dt = \int_0^1 51t^2 - 14t - 1\, dt \\ &= 17t^3 - 7t^2 - t\Big|_0^1 = 9\end{aligned}$$

b) $F(x,y) = \langle e^x, e^y\rangle$ and C is the top half of the unit circle from $(1,0)$ to $(-1,0)$.

Solution: First, we can check whether this is conservative. If we calculate $Q_x = 0$ and $P_y = 0$, since the domain is everything, we can conclude it is conservative. So we can find a potential function $f(x,y) = e^x + e^y$. Then $\int_C \vec{F} \circ d\vec{r} = f(\text{end}) - f(\text{start})$, so we get our answer of $f(-1,0) - f(1,0) = e^-1 + e^0 - e^1 - e^0 = \frac{1}{e} - e$.

We can double check the brute force way gives the same answer. We want to parameterize the curve connecting our two points. We can take the standard $\vec{r}(t) = \langle \cos t, \sin t\rangle$ where t goes from 0 to π. Then we want to calculate $d\vec{r} = \vec{r'}(t)\, dt = \langle -\sin t, \cos t\rangle\, dt$. Now, we can start putting everything together in our integral.

$$\begin{aligned}\int_C \langle e^x, e^y\rangle \circ d\vec{r} &= \int_0^\pi \langle e^{\cos t}, e^{\sin t}\rangle \circ \langle -\sin t, \cos t\rangle\, dt \\ &= \int_0^\pi -\sin t e^{\cos t} + \cos t e^{\sin t}\, dt \\ &= e^{\cos t} + e^{\sin t}\Big|_0^\pi = e^{-1} + e^0 - e^1 - e^0 = \frac{1}{e} - e\end{aligned}$$

c) $F(x,y,z) = \langle y, z, x\rangle$ and C is the intersection of the surfaces $z = x^2 + y^2$ and $x = y$ from the origin to $(1,1,2)$.

Solution: Again we start by trying to see if it is conservative. Calculating the curl gives $\nabla \times \vec{F} = \langle -1,-1,-1\rangle$, so it is not conservative, and we cannot use FTL. Instead, we start by parameterizing the curve. This can be accomplished by $\vec{r}(t) = \langle t, t, 2t^2\rangle$ where t goes from 0 to 1. Then $d\vec{r} = \langle 1,1,4t\rangle\, dt$, and we can put everything together:

$$\begin{aligned}\int_C \langle y,z,x\rangle \circ d\vec{r} &= \int_0^1 \langle t, 2t^2, t\rangle \circ \langle 1,1,2t^2\rangle\, dt \\ &= \int_0^1 t + 2t^2 + 2t^3\, dt \\ &= \frac{t}{2} + \frac{2t^3}{3} + \frac{2t^4}{4}\Big|_0^1 = \frac{5}{3}\end{aligned}$$

We could also use Stokes' Theorem to evaluate this if we want to test our understanding of Stokes Theorem.

4. A particle travels along the level curve C given by $\sin x + e^y = 7$ from the point $(0, \ln 7)$ to $\left(\frac{\pi}{2}, \ln 6\right)$. Find $\int_C \cos x\, dx + e^y\, dy$.

Solution: Again, the first thing to check is whether our vector function $\vec{F}(x,y) = \langle \cos x, e^y \rangle$ is conservative. It is in fact, and we can find a potential function $f(x,y) = \sin x + e^y$. Then we can use the Fundamental Theorem of line Integrals to get $\int_C \vec{F} \circ d\vec{r} = f\left(\frac{\pi}{2}, \ln 6\right) - f(0, \ln 7) = (1+6) - (0+7) = 0$.

5. The height of a mountain is described by $h(x,y) = x^2y + e^{xy} + x$. Suppose some walks along the path C from $(0,0)$ to $(2,2)$ determined by making sure at every point, the person is moving in the direction where the slope is the steepest. What is $\int_C \nabla h \circ d\vec{r}$?

   Solution: The description of the path the person walks is a red herring. We are integrating ∇h, which is obviously conservative – it is the gradient of h. So all we care about are the end points of the path. Our answer is just $h(2,2) - h(0,0) = 8 + e^4 + 2 - 0 - 1 - 0 = 9 + e^4$.

0.5 Green's Theorem

1. True/False practice:
   a) If the domain of $\vec{F}(x,y)$ has a hole at the origin, then Green's Theorem will fail for any region in the plane.

      Solution: False. We can still use Green's Theorem for any region that does not contain the origin (the hole).

   b) A positively-oriented curve is one that travels counterclockwise in the plane.

      Solution: False. This is sometimes true, but not always. When traveling along the boundary of a region, being positively-oriented means that the region is on our left.

2. Suppose C is the triangular path moving from $(0,0)$ to $(2,1)$ to $(1,2)$ and then back to the origin. Using Green's Theorem, find $\int_C \langle e^x \sin x + y, y^2 - x^2 \rangle \circ d\vec{r}$.

   Solution: Green's theorem let's us calculate this line integral by working with the double integral of $Q_x - P_y$. So first we want to calculate these partial derivatives $Q_x - P_y = -2x - 1$. So all we need to calculate is $\iint_R (-2x-1)\,dA$ where R is the inside of the triangle. Note that the sign of this integral is correct since the curve is positively-oriented (the region is the the left of someone walking along the original path).

   To evaluate this integral, we need to break the region into two pieces based on which line is the top part of our triangle. This gives us:

$$\begin{aligned}
\iint_R (-2x-1)\,dA &= \int_0^1 \int_{\frac{x}{2}}^{2x} (-2x-1)\,dy\,dx + \int_1^2 \int_{\frac{x}{2}}^{3-x} (-2x-1)\,dy\,dx \\
&= \int_0^1 -2xy - y \Big|_{\frac{x}{2}}^{2x} dx + \int_1^2 -2xy - y \Big|_{\frac{x}{2}}^{3-x} dx \\
&= \int_0^1 -4x^2 - 2x + x^2 + \frac{x}{2}\,dx + \int_1^2 -2x(3-x) - (3-x) + x^2 + \frac{x}{2}\,dx \\
&= -\frac{4}{3} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{11}{4} = -\frac{9}{2}
\end{aligned}$$

3. Find $\int_C x^2y\,dx - xy^2\,dy$ where C is the circle of radius 2 centered at the origin traveling clockwise.

Solution: Again using Green's Theorem, $Q_x - P_y = -y^2 - x^2$, so we are taking $\iint_R -y^2 - x^2\, dA$ where R is the disc of radius 2 centered at the origin. However, it is important to consider the orientation of the boundary of this region, i.e. the original curve. Since C was moving clockwise, the region is to the right of this path. This means our original curve actually has negative orientation. We can fix this by simply multiplying through by -1, so we get:

$$-\int_C \vec{F} \circ d\vec{r} = \iint_R Q_x - P_y\, dA$$

Then we just need to evaluate $\iint_R y^2 + x^2\, dA$. This is most easily done by changing to polar coordinates. So $y^2 + x^2 = r^2$, $da = r\, dr\, d\theta$ and the region becomes $0 \le \theta \le 2\pi$ and $0 \le r \le 2$. Then our final integral is just $\int_0^{2\pi} \int_0^2 r^3\, dr\, d\theta = 2\pi \cdot 4$.

4. Suppose R is the region between $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$. Use Green's Theorem to find $\iint_R 2y\, dA$.

   Solution: We can use Green's Theorem in the other direction to find a double integral of a region using a line integral over the boundary. In order to do this, we need to find some $\vec{F}(x, y) = \langle P, Q\rangle$ such that $Q_x - P_y = 2y$. There are many choices that we could take here, but let's try something like $\vec{F}(x, y) = \langle 0, 2yx\rangle$. Then we are trying to compute $\int_C \vec{F} \circ d\vec{r}$, where C is the boundary of our original region.

   We will have to break this boundary into two pieces. First we could travel along the line $y = 0$ from the origin to $(\pi, 0)$. Then we could travel back along the curve $y = \sin x$ to the origin. For the first line integral, we can parameterize the curve by $\vec{r}(t) = \langle t, 0\rangle$ where t goes from 0 to π; then $\vec{r'}(t) = \langle 1, 0\rangle$.

$$\int_{C_1} \vec{F} \circ d\vec{r} = \int_0^\pi \langle 0, 2yx\rangle \circ \langle 1, 0\rangle\, dt = 0$$

   The second piece of this boundary can be parameterized by $\vec{r}(t) = \langle t, \sin t\rangle$ where t goes from π to 0. (Note the direction that it travels.) Then we get $\vec{r'}(t) = \langle 1, \cos t\rangle$, and our integral is:

$$\begin{aligned}\int_{C_2} \vec{F} \circ d\vec{r} &= \int_\pi^0 \langle 0, 2t\sin t\rangle \circ \langle 1, \cos t\rangle\, dt \\ &= \int_\pi^0 2t\cos t \sin t\, dt\end{aligned}$$

   We will have to use some integration by parts to evaluate this. Let $u = t$ and $dv = 2\sin t\cos t\, dt$; then we get $du = dt$ and $v = \sin^2 t$. So integration by parts gives us:

$$\begin{aligned}\int_\pi^0 2t\cos t\sin t\, dt &= t\sin^2 t\Big|_\pi^0 - \int_\pi^0 \sin^2 t\, dt \\ &= 0 - \int_\pi^0 \frac{1}{2}(1 - \cos 2t)\, dt \\ &= \frac{\pi}{2}\end{aligned}$$

5. Suppose C is the ellipse given by $25x^2 + 9y^2 = 1$ and $\vec{F}(x, y) = \langle x\ln(x^2 + y^2), y\ln(x^2 + y^2)\rangle$. (Hint: It'd be much easier to integrate this over the unit circle than the ellipse...)

Solution: If we try to use Green's Theorem, we get that $Q_x - P_y = \frac{2xy}{x^2+y^2} - \frac{2xy}{x^2+y^2} = 0$. However, this does not mean that our integral must be zero. The issue is that our vector function is undefined at the origin, so there is a hole; and we cannot use Green's theorem over a region with a hole in it. Instead let C be our original ellipse oriented counterclockwise and D the unit circle oriented clockwise. Then if R is the region between C and D, we can almost use Green's Theorem just fine. With the orientations specified, the region is to the **right** of both of our new boundary curves. So this will introduce a negative sign into Green's Theorem to get the correct orientation.

$$0 = -\iint_R Q_x - P_y \, dA = \int_C \vec{F} \circ d\vec{r} + \int_D \vec{F} \circ d\vec{r}$$

So we just need to calculate $\int_D \vec{F} \circ d\vec{r}$ to get our final answer. To do this, parameterize our curve as $\vec{r}(t) = \langle \cos t, \sin t \rangle$ where t goes from 2π to 0. (Why am I doing it in this direction? Why did I want D to be clockwise?) Then when we start to evaluate $F(x, y)$ along this curve, we get $F(\cos t, \sin t) = \langle \sin t, \ln(1), \cos t \ln(1) \rangle = \langle 0, 0 \rangle$. So in fact, this line integral is 0. This means our original answer was 0. But again, this is just luck, and we aren't guaranteed this even though $Q_x - P_y = 0$, since there is a hole.

0.6 Surface Integrals

1. Suppose S is the triangle formed by the plane $x + y + z = 1$ between the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Using a double integral, find the area of this surface.

   Solution: The surface area of S is given by $\iint_S 1 \, dS$. The first step in calculating this is to parameterize the surface. Since the surface is given by the function $f(x, y) = 1 - x - y$, we can do this easily by $\vec{r}(s, t) = \langle s, t, 1 - s - t \rangle$. In order to complete this parameterization, we need to know the domain for s and t. In this case, we will have the triangle $s, t \geq 0$ and $s + t \leq 1$ as the domain of our parameters; call this region R.

   Then to translate dS into these new parameters, we need to know the area of the parallelograms that arise when we make this shift. This can be done by calculating the length of $\vec{r}_s \times \vec{r}_t$, the cross product of the tangent vectors for our parameterization.

   $$|\vec{r}_s \times \vec{r}_t| = |\langle 1, 0, -1 \rangle \times \langle 0, 1, -1 \rangle| = |\langle 1, 1, 1 \rangle| = \sqrt{3}$$

   Then we can set up our integral, where R is the region in the st-plane the bounds our parameters:

   $$\iint_S 1 \, dS = \iint_R 1\sqrt{3} \, dA$$

   So this is just $\sqrt{3}$ times the area of R. But R is the right triangle with base and height of 1, so our final answer is $\frac{\sqrt{3}}{2}$.

2. Suppose E is the solid formed by intersecting the cylinder $x^2 + y^2 \leq 1$ with the planes $z = 0$ and $z = x + y + 5$. Find the total surface area of this solid.

   Solution: There are three pieces to this surface area. We have S_1, the base when $z = 0$, which is just a disc of radius 1, so it has an area of π. We have S_2 the top of this cylinder when $z = x + y + 5$, and then we have S_3, the outside part of the cylinder.

To find the area of S_2, we first parameterize the surface. Since the surface is given by a function, we can parameterize this by $\vec{r}(s,t) = \langle s, t, 5+s+t \rangle$, where $s^2+t^2 \leq 1$ is the region R for our parameters. Then we want to calculate the length of $\vec{r}_s \times \vec{r}_t$ to adjust the integral for this parameterization.

$$|\vec{r}_s \times \vec{r}_t| = |\langle 1,0,1 \rangle \times \langle 0,1,1 \rangle| = |\langle -1,-1,1 \rangle| = \sqrt{3}$$

Then we can set up our integral, where R is the region in the st-plane the bounds our parameters:

$$\iint_{S_2} 1\, dS = \iint_R 1\sqrt{3}\, dA = \sqrt{3}\pi$$

To find the area of S_3, we again need to parameterize our surface. However, the surface is no longer the graph of a function, so we can't be as straightforward as before. Instead, since we are looking at the surface of a cylinder, we can let $\vec{r}(s,t) = \langle \cos t, \sin t, s \rangle$ where t traces out a circle from 0 to 2π. Then the bounds for s are given by $0 \leq s \leq 5+\cos t+\sin t$. Then we want to calculate the length of $\vec{r}_s \times \vec{r}_t$ to adjust the integral for this parameterization.

$$|\vec{r}_s \times \vec{r}_t| = |\langle 0,0,1 \rangle \times \langle -\sin t, \cos t, 0 \rangle| = |\langle -\cos t, -\sin t, 0 \rangle| = 1$$

Then we can set up our integral, where R is the region in the st-plane the bounds our parameters:

$$\begin{aligned}\iint_{S_3} 1\, dS &= \iint_R 11\, dA \\ &= \int_0^{2\pi} \int_0^{5+\cos t+\sin t} 1\, ds\, dt \\ &= \int_0^{2\pi} 5+\cos t+\sin t\, dt = 10\pi\end{aligned}$$

Finally we have to remember to put everything together. The surface area of all three of our pieces together is $\pi + \sqrt{3}\pi + 10\pi$.

3. Find the surface area of the paraboloid $z = x^2+y^2$ inside the cylinder $x^2+y^2 = 9$.

   Solution: Like always, we start by parameterizing our surface. This time, it is the graph of a function, so we can take $\vec{r}(s,t) = \langle s, t, s^2+t^2 \rangle$. The fact that we are inside the cylinder of radius 3 tells us that our parameters are bounded by $s^2+t^2 \leq 9$, this is the region R that we are integrating s and t in.

   Next, in order to change dS into these parameters, we need the length of $\vec{r}_s \times \vec{r}_t$.

$$|\vec{r}_s \times \vec{r}_t| = |\langle 1,0,2s \rangle \times \langle 0,1,2t \rangle| = |\langle -2s,-2t,1 \rangle| = \sqrt{1+4s^2+4t^2}$$

   So we can set up our integral in terms of our new parameters:

$$\iint_S 1\, dS = \iint_R 1\sqrt{1+4s^2+4t^2}\, dA$$

   The easiest way to integrate this now is with polar coordinates $s = r\cos\theta$ and $t = r\sin\theta$ where $0 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$. Then additionally, $dA = r\, dr\, d\theta$, giving us:

$$\iint_R 1\sqrt{1+4s^2+4t^2}\, dA = \int_0^{2\pi} \int_0^3 \sqrt{1+4r^2} r\, dr\, d\theta$$

Now we use u-substitution to evaluate this. Let $u = 4r^2$, so that $du = 8r\,dr$. Then we get:

$$\int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}r\,dr\,d\theta = 2\pi\int_0^{36}\frac{1}{8}\sqrt{1+u}\,du = 2\pi\frac{1}{12}(37^{\frac{3}{2}}-1)$$

4. Suppose S is the triangle formed by the plane $x+y+z=1$ between the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, oriented so that normal vectors point upwards.
   a) Find $\iint_S xy\,dS$.

   Solution: Like always, we paramterize. Although this was already done in the first question in this section.

$$\vec{r}(s,t) = \langle s, t, 1-s-t\rangle \text{ and } s,t \geq 0 \text{ and } s+t \leq 1$$

   Then once again, our scaling factor with this parameterization is $dS = \sqrt{3}\,dA$ (see above). So we can start plugging stuff in:

$$\begin{aligned}\iint_S xy\,dS &= \int_0^1\int_0^{1-s} st\sqrt{3}\,dt\,ds\\ &= \int_0^1 \frac{\sqrt{3}}{2}st^2\Big|_0^{1-s} ds\\ &= \int_0^1 \frac{\sqrt{3}}{2}s(1-s)^2\,ds = \frac{\sqrt{3}}{24}\end{aligned}$$

   b) Find $\iint_S \langle x,y,z\rangle \circ d\vec{S}$.

   Solution: For surface integrals of vector functions, we still want to start with a parameterization of our surface.

$$\vec{r}(s,t) = \langle s, t, 1-s-t\rangle \text{ and } s,t \geq 0 \text{ and } s+t \leq 1$$

   But now, we translate $d\vec{S}$ slightly differently; instead of taking the length of the normal vector, we just use the normal vector in the dot product $d\vec{S} = \vec{n}\,dA$. We calculated above the normal vector $\vec{n} = \langle 1,1,1\rangle$. Note that this is in fact pointing upwards like we wanted. Now we have everything we need to solve the integral:

$$\begin{aligned}\iint_S \langle x,y,z\rangle \circ d\vec{S} &= \iint_R \langle s,t,1-s-t\rangle \circ \langle 1,1,1,\rangle\,dA\\ &= \int_0^1\int_0^{1-s} s+t+1-s-t\,dt\,ds\\ &= \int_0^1\int_0^{1-s} 1\,dt\,ds = \frac{1}{2}\end{aligned}$$

5. Suppose the surface S is the top half of the unit sphere centered at the origin, oriented inwards.
   a) If the density is given by $f(x,y,z) = 2x+3y+z$, find the total mass of this surface.

   Solution: We are trying to find $\iint_S f(x,y,z)\,dS$. First parameterize. There are lots of ways to parameterize the top half of a sphere. It is a function, so we can use our basic trick of letting $x = s$ and $y = t$. But it will be easier to go straight to spherical coordinates here. This will give us:

$$\vec{r}(s,t) = \langle \cos t \sin s, \sin t \sin s, \cos s \rangle$$

Here $0 \leq t \leq 2p$ are our "longitude lines", and $0 \leq s \leq \frac{\pi}{2}$ ranges from the north pole to the equator. Now we calculate the length of the normal vector to scale dS:

$$\begin{aligned}|\vec{r}_s \times \vec{r}_t| &= |\langle \cos t \cos s, \sin t \cos s, -\sin s\rangle \times \langle -\sin t \sin s, \cos t \sin s, 0\rangle| \\ &= \left|\langle \sin^2 s \cos t, \sin^2 s \sin t, \cos s \sin s\rangle\right| = \sqrt{\sin^4 s + \cos^2 s \sin^2 s} = |\sin s|\end{aligned}$$

Since $0 \leq s \leq \frac{\pi}{2}$, we already have $\sin s \geq 0$, so we can drop the absolute value signs. Then we can start plugging into our integral:

$$\begin{aligned}\iint_S 2x + 3y + z\, dS &= \iint_R (2\cos t \sin s + 3 \sin t \sin s + \cos s) \sin s\, dA \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 2\cos t \sin^2 s + 3 \sin t \sin^2 s + \cos s \sin s\, ds\, dt \\ &= 0 + 0 + 2\pi \cdot \frac{1}{2}\end{aligned}$$

(Evaluate the first two terms in the integral with respect to t first.)

b) If $\vec{F}(x,y,z) = \langle x^2, 0, y^2\rangle$, find the flux of $\vec{F}$ across S.

Solution: We still want a parameterization and we can use the one above:

$$\vec{r}(s,t) = \langle \cos t \sin s, \sin t \sin s, \cos s \rangle$$

Where $0 \leq t \leq 2\pi$ and $0 \leq s \leq \frac{\pi}{2}$. Then we have also calculated the normal vector already, $\vec{n}(s,t) = \langle \sin^2 s \cos t, \sin^2 s \sin t, \cos s \sin s$. Before we continue, we must worry about the *orientation* of our surface. Originally, we had wanted the normal vectors for our circle to point inside the sphere. If we look at some point on our sphere, like the Cartesian point $(1,0,0)$, i.e. the spherical point $t = 0$ and $s = \frac{\pi}{2}$, we have a normal vector $\langle 1,0,0\rangle$. This is currently pointing out of the sphere, so we should correct this by negating our normal vector:

$$\vec{n}(s,t) = \langle -\sin^2 s \cos t, -\sin^2 s \sin t, -\cos s \sin s\rangle$$

Now we can finally plug everything into our giant integral:

$$\begin{aligned}\iint_S \langle x^2, 0, y^2\rangle \circ d\vec{S} &= \iint_R \langle \cos^2 t \sin^2 s, 0, \sin^2 t \sin^2 s\rangle \circ \langle -\sin^2 s \cos t, -\sin^2 s \sin t, -\cos s \sin s\rangle\, dA \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} -\cos^3 t \sin^4 s - \cos s \sin^2 t \sin^3 s\, ds\, dt \\ &= -\int_0^{2\pi} \cos^3 t\, dt \int_0^{\frac{\pi}{2}} \sin^4 s\, ds - \int_0^{2\pi} \sin^2 t\, dt \int_0^{\frac{\pi}{2}} \cos s \sin^3 s\, ds \\ &= 0 - \pi \cdot \frac{1}{4}\end{aligned}$$

0.7 Gradient, Curl, Divergence

1. True/False practice:
   a) The curl of the curl of $\vec{F}$ is always $\langle 0,0,0\rangle$.

   Solution: False. There are many functions where we can just calculate $\nabla \times (\nabla \times \vec{F})$ and get something nonzero. For instance, we could start with $\vec{F}(x,y,z) = \langle x^2, x^2, x^2\rangle$, we get the curl of the curl of $\vec{F}$ equals $\langle -2,-2,0\rangle$.

   b) The curl of the curl of $\vec{F}$ is never $\langle 0,0,0\rangle$.

   Solution: False. There are also functions where we *do* get the zero vector. For instance, starting with $\vec{F}(x,y,z) = \langle x,y,z\rangle$ gives us $\nabla \times (\nabla \times \vec{F}) = \langle 0,0,0\rangle$. (It is interesting to try to determine which functions give us the zero vector and which give something nonzero).

2. For the following vector fields, compute the curl and the divergence. Determine whether $\vec{F}$ is the gradient of some f, and find such a function if possible. Determine whether $\vec{F}$ is the curl of some function $\vec{G}$, and find such a function if possible.
   a) $F(x,y,z) = \langle x^2, y^2, z^2\rangle$

   Solution: We can calculate the curl by setting up a 3×3 array:

$$\nabla \times \vec{F} = \begin{vmatrix} 1^{st} & 2^{nd} & 3^{rd} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = \langle 0,0,0\rangle$$

   The divergence is much easier to calculate: $\nabla \circ \vec{F} = 2x + 2y + 2z$.

   Since the curl is the zero vector and the domain of our vector function is everything, we can conclude that $\vec{F}$ is the gradient of some function. By taking a couple partial antiderivatives, we can see that $f(x,y,z) = \frac{1}{3}(x^3 + y^3 + z^3)$ works. Since the divergence is not zero, this is not the curl of some function.

   b) $F(x,y,z) = \langle \sin y, \sin z, \sin x\rangle$

   Solution: We can calculate the curl by setting up a 3×3 array:

$$\nabla \times \vec{F} = \begin{vmatrix} 1^{st} & 2^{nd} & 3^{rd} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & \sin z & \sin x \end{vmatrix} = \langle -\cos z, -\cos x, -\cos y\rangle$$

   The divergence is much easier to calculate: $\nabla \circ \vec{F} = 0 + 0 + 0$.

   Since the curl is not the zero vector, this is not the gradient of any function. However, the divergence is zero, and since the domain of our vector function is all of $\mathbb{R}^3$ (i.e. the domain is simply connected), this is the curl of something. So we can find some $\vec{G} = \langle P,Q,R\rangle$ so that $\vec{F} = \nabla \times \vec{G}$.

We can do this by setting up a system of equations from the formula for curl:

$$R_y - Q_z = \sin y$$
$$P_z - R_x = \sin z$$
$$Q_x - P_y = \sin x$$

Then we can work through by guess and checking some antiderivatives. For instance, we would make the first one true if $R = -\cos y$ and $Q_z = 0$. Then $P = -\cos z$ and $Q = -\cos x$ will satisfy the rest of the equations.

c) $F(x, y, z) = \langle xyz, xyz, xyz\rangle$

Solution: We can calculate the curl by setting up a 3×3 array:

$$\nabla \times \vec{F} = \begin{vmatrix} 1^{st} & 2^{nd} & 3^{rd} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & xyz & xyz \end{vmatrix} = \langle xz - xy, yx - yz, zy - zx\rangle$$

The divergence is much easier to calculate: $\nabla \circ \vec{F} = yz + xz + xy$.

Since the curl is not the zero vector, this is not a conservative vector function. Since the divergence is not zero, this is not the curl of something.

d) $F(x, y, z) = \langle 1, 1, 1\rangle$

Solution: The curl will be $\langle 0, 0, 0\rangle$. The divergence will be $0 + 0 + 0 = 0$. So this is the gradient of something, as the domain is all of $\mathbb{R}^3$. For instance we could take $f(x, y, z) = x + y + z$ as our potential function. It is also the curl of something since the divergence is zero. To find the "anti-curl", we can solve the equations:

$$R_y - Q_z = 1$$
$$P_z - R_x = 1$$
$$Q_x - P_y = 1$$

This can be accomplished with $\vec{G}(x, y, z) = \langle z, x, y\rangle$.

3. Show that $\nabla \circ (\vec{F} \times \vec{G}) = (\nabla \times \vec{F}) \circ \vec{G} - \vec{F} \circ (\nabla \times \vec{G})$.

Solution: Let $\vec{F}(x, y, z) = \langle P, Q, R\rangle$ and $\vec{G}(x, y, z) = \langle A, B, C\rangle$. Then first, we can calculate $\vec{F} \times \vec{G} = \langle QC - RB, RA - PC, PB - QA\rangle$. Then the left side of our equation is:

$$\begin{aligned} \nabla \circ (\vec{F} \times \vec{G}) &= (QC - RB)_x + (RA - PC)_y + (PB - QA)_z \\ &= Q_xC + QC_x - R_xB - RB_x + R_yA + RA_y - P_yC - PC_y + P_zB + PB_z - Q_zA - QA_z \\ &= [(R_y - Q_z)A + (P_z - R_x)B + (Q_x - P_y)C] - [(C_y - B_z)P + (A_z - C_x)Q + (B_x - A_y)R] \end{aligned}$$

Where first we just used the basic product rule for derivatives a lot of times, and then we grouped together common terms. Then the first three terms give $(\nabla \times \vec{F}) \circ \vec{G}$ and the last three terms give $\vec{F} \circ (\nabla \times \vec{G})$.

4. Show that *any* continuous scalar function $f(x, y, z)$ is equal to the divergence of some vector field.

Solution: We don't have to work to hard to find some $\vec{F}(x, y, z) = \langle P, Q, R\rangle$ that has $P_x + Q_y + R_z = f(x, y, z)$. For instance, let $Q = R = 0$. Then we could take $P = \int_0^x f(t, y, z)\, dt$.

5. Suppose $f(x,y)$ and $g(x,y)$ have continuous partial derivatives, C is the unit circle oriented counterclockwise, and D is the region inside the unit circle. If $\vec{n}$ is the unit normal vector for the unit circle, using Green's Theorem, it is possible to prove that

$$\int_C f \cdot (\nabla g \circ \vec{n})\ ds = \iint_D \nabla f \circ \nabla g \, dA + \iint_D f \cdot \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \right)\ dA.$$

Suppose $h(x,y)$ is a harmonic function, and $h(x,y) = 0$ for any point on the unit circle. Use the above fact to show that $\iint_D |\nabla h|^2 \, dA = 0$.

Solution: We want to use the given equation for $f = g = h$. Then just plugging these letters in everywhere gives:

$$\int_C h \cdot (\nabla h \circ \vec{n})\ ds = \iint_D \nabla h \circ \nabla h \, dA + \iint_D h \cdot \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right)\ dA.$$

On the left, the line integral will equal 0. This is because $h = 0$ at any point on C, so the function $h \cdot (\nabla h \circ \vec{n})$ will equal 0 along the curve C. The first double integral is what we are trying to solve for $\iint_D |\nabla h|^2 \, dA$. The final term, the second double integral is also 0. This is because h is a *harmonic function* which means that if we plug it into the given differential equation, we will get 0. So all together we have $0 = \iint_D |\nabla h|^2 \, dA + 0$, which proves what we wanted.

0.8 Stokes' Theorem and the Divergence Theorem

1. Use Stokes' Theorem to evaluate the line integral $\int_C \vec{F} \circ d\vec{r}$ where C is the triangle from $(1,0,0)$ to $(0,1,0)$ to $(0,0,1)$ to $(1,0,0)$ and $\vec{F}(x,y,z) = \langle x + y^2, y + z^2, z + x^2 \rangle$.

Solution: Stokes' Theorem says that we can evaluate this line integral by calculating the double integral of the curl of $\vec{F}$ on a region that is bounded by C. For the region R, we can just take the flat triangle in $\mathbb{R}^3$ whose corners are those three points. We should double check the positive orientation. As we move around the three points in the given order, the triangle is always to our left, so the orientation of our surface will have normal vectors pointing up.

Then we need to calculate the curl of $\vec{F}$. This gives us $\nabla \times \vec{F} = \langle -2z, -2x, -2y \rangle$. Then we calculate the surface integral $\iint_R \langle -2z, -2x, -2y \rangle \circ d\vec{S}$, as before. We've parameterized this surface in previous problems:

$$\vec{r}(s,t) = \langle s, t, 1 - s - t \rangle \text{ and } s, t \geq 0 \text{ and } s + t \leq 1$$

Then the normal vector is given by the cross product of the partial derivatives:

$$\vec{r}_s \times \vec{r}_t = \langle 1, 0, -1 \rangle \times \langle 0, 1, -1 \rangle = \langle 1, 1, 1 \rangle$$

Double check that these vectors have the correct orientation. We wanted the normal vectors to point up from the surface, and they do. So now we can plug everything in:

$$\begin{aligned}\int_C \vec{F} \circ d\vec{r} &= \iint_R \langle -2(1-s-t), -2s, -2t\rangle \circ \langle 1,1,1\rangle\, dA \\ &= \int_0^1 \int_0^{1-s} -2 + 2s + 2t - 2s - 2t\, dt\, ds \\ &= -2 \iint_R 1\, dA = -1\end{aligned}$$

Where I evaluated the last integral by noting that R was a right triangle in the plane, so I just wanted its area.

2. Let $\vec{G} = \langle z, x, -x\rangle$ and S is the surface of the paraboloid $z = x^2 + y^2$ from $z = 0$ to $z = 9$ oriented so that the normal vectors point outside.
   a) What is the positively-oriented boundary of S?

      Solution: The boundary of S is the circle given by $x^2 + y^2 = 9$ on the plane $z = 9$. We just need to decide whether it travels clockwise or counterclockwise when looking down on the circle form above. The normal vectors of our surface point outwards, and positive orientation of this boundary means that if we walk along the boundary, with our head in the direction of the normal vector, then the region is on our left. The only way this is true is if we are traveling around the circle clockwise when viewed from above.
   b) Show that $\vec{G}$ is the curl of something.

      Solution: We can do this by calculating the divergence, since the domain of the given function is all of $\mathbb{R}^3$. Since the divergence is in fact 0, we can find some $\vec{F}$ so that $\nabla \times \vec{F} = \vec{G}$. We can find such a $\vec{F}$ by solving these equations:

$$\begin{aligned}R_y - Q_z &= z \\ P_z - R_x &= x \\ Q_x - P_y &= -x\end{aligned}$$

      Working one equation at a time, first make a guess that $R = yz$ and $Q_z = 0$. Then we'd need $P = xz$ for the second equation. And finally, we'd get $Q = -\frac{x^2}{2}$. So $\vec{G}$ is the curl of $\vec{F} = \langle xz, -\frac{x^2}{2}, yz\rangle$.
   c) Use Stokes' Theorem to calculate $\iint_S \vec{G} \circ d\vec{S}$.

      Solution: Since $\vec{G}$ is the curl of something, Stokes' Theorem says that instead of calculating the surface integral of $\vec{G}$ directly, we can use a line integral along the boundary.

$$\iint_S \vec{G} \circ d\vec{S} = \int_C \vec{F} \circ d\vec{r}$$

      We found C in the first part as the clockwise circle. This could be parameterized by $\vec{r}(t) = \langle 3\cos t, 3\sin t, 9\rangle$ where t goes from 2π to 0 (so it moves clockwise). We found

$\vec{F} = \langle xz, -\frac{x^2}{2}, yz\rangle$ in the second part. Then we can put it all together:

$$\begin{aligned}\int_C \vec{F}\circ d\vec{r} &= \int_{2\pi}^{0} \langle 27\cos t, -\frac{9\cos^2 t}{2}, 27\sin t\rangle \circ \langle -3\sin t, 3\cos t, 0\rangle\, dt\\ &= \int_{2\pi}^{0} -27\sin t\cos t - \frac{9\cos^3 t}{2}\, dt\\ &= \int_{0}^{2\pi} 27\sin t\cos t + \frac{9\cos^3 t}{2}\, dt\\ &= 9\cdot 0 + 0 = 0\end{aligned}$$

3. Let $\vec{F} = \langle y+z, 2x, 2\rangle$ and E is the solid given by the cone $y^2 = x^2+z^2$ between $y=0$ and $y=4$.
   a) What is the positively-oriented boundary of E?

      Solution: The boundary consists of two surfaces. We have the outside of the cone S_1 given by $y^2 = x^2+z^2$ from $y=0$ to $y=4$. But then we also have the top of the cone S_2 given by the disc $x^2+z^2 = 4$ on the plane $y=4$. For positive orientation, we want to make sure all of the vectors are pointing out of the surface.
   b) If S is this boundary, calculate $\iint_S \vec{F}\circ d\vec{S}$ directly.

      Solution: We have to break this into two integrals, and parameterize them separately. On S_1, the cone is a function of x and z, so we can parameterize this by:

$$\vec{r}(s,t) = \langle s, \sqrt{s^2+t^2}, t\rangle$$

Where $s^2+t^2 \le 4$ is our R for the parameters. Next we calculate the normal vector:

$$\vec{n}(s,t) = \vec{r}_s\times\vec{r}_t = \left\langle 1, \frac{s}{\sqrt{s^2+t^2}}, 0\right\rangle \times \left\langle 0, \frac{t}{\sqrt{s^2+t^2}}, 1\right\rangle = \left\langle \frac{s}{\sqrt{s^2+t^2}}, -1, \frac{t}{\sqrt{s^2+t^2}}\right\rangle$$

Here we double check that the normal vectors are correctly oriented. Since $y=-1$ for all of these vectors, these are indeed pointing out of the cone (draw a picture). Then we have the integral:

$$\begin{aligned}\iint_{S_1}\langle y+z, 2x, 2\rangle\circ d\vec{S} &= \iint_R \left\langle \sqrt{s^2+t^2}+t, 2s, 2\right\rangle \circ \left\langle \frac{s}{\sqrt{s^2+t^2}}, -1, \frac{t}{\sqrt{s^2+t^2}}\right\rangle dA\\ &= \iint_R s + \frac{st}{\sqrt{s^2+t^2}} - 2s + 2\frac{t}{\sqrt{s^2+t^2}}\, dA\end{aligned}$$

Then the easiest way to evaluate this is with polar coordinates. Let $s = r\cos\theta$ and $t = r\sin\theta$, so our region is just $0\le\theta\le 2\pi$ and $0\le r\le 2$, and $dA = r\,dr\,d\theta$. Then this becomes:

$$\iint_S \vec{F}\circ d\vec{S} = \int_0^{2\pi}\int_0^2 (r\cos\theta + r\sin\theta\cos\theta - 2r\cos\theta + 2\sin\theta)\, r\,dr\,d\theta = 0$$

Then we want to look at the top of the cone, S_2. We can parameterize this by $\vec{r}(s,t) = \langle s\cos t, 4, s\sin t$ where $0\le s\le 2$ and $0\le t\le 2\pi$. Then we calculate the normal vector:

$$\vec{n}(s,t) = \vec{r}_s\times\vec{r}_t = \langle\cos t, 0, \sin t\rangle\times\langle -s\sin t, 0, s\cos t\rangle = \langle 0, -s(\sin^2 t + \cos^2 t), 0\rangle = \langle 0, -s, 0\rangle$$

Now we double check that it is correctly oriented. It is suppose to point out of the cone, but this is pointing in the negative y-direction, which is inside the cone. So we just fix that by flipping all of the normal vectors by negative 1 and work with $\vec{n} = \langle 0, s, 0\rangle$. Then we can set up the integral:

$$\begin{aligned}\iint_{S_2} \langle y+z, 2x, 2\rangle \circ d\vec{S} &= \iint_R \langle 4 + s\sin t, 2s\cos t, 2\rangle \circ \langle 0, s, 0\rangle \, dA \\ &= \int_0^{2\pi}\int_0^2 2s^2 \cos t \, ds\, dt = 0\end{aligned}$$

So all together, this surface integral is 0.

c) Use the Divergence Theorem to calculate $\iint_S \vec{F} \circ d\vec{S}$.

Solution: The divergence theorem tells us that we can calculate a surface integral by instead working with the triple integral over the solid enclosed by our surface. So in particular, if E is the solid cone bounded by S:

$$\iint_S \vec{F} \circ d\vec{S} = \iiint_E \text{ div } \vec{F} \, dV$$

So first let us calculate the divergence of $\vec{F}$. This gives $\nabla \circ \vec{F} = 0 + 0 + 0 = 0$. So the divergence theorem says that:

$$\iint_S \vec{F} \circ d\vec{S} = \iiint_E 0 \, dV = 0$$

This matches what we found in the previous part, so the universe still works.